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Bifurcations from locally to globally riddled basins

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We give sufficient conditions for the occurrence of locally and globally riddled basins in coupled systems. Two different types of bifurcations from locally to globally riddled basins are described.
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The phenomenon of riddled basins [1] has become an important focus in the study of nonlinear dynamics. These studies are important not only from a theoretical point of view, leading to better understanding of chaotic systems, but from the point of view of possible applications as well. Riddling typically occurs in chaotic systems with symmetric invariant manifolds, which are often encountered in chaos synchronization schemes [2], particularly those used for secure communication [3]. In such systems riddling can lead to the loss of synchronization.

Consider two identical chaotic systems, $x_{n+1}=f(x_n)$ and $y_{n+1}=f(y_n)$, $x,y\in\mathbb{R}$, evolving on an *asymptotically stable chaotic attractor* A (the set A is an asymptotically stable attractor if for any sufficiently small neighborhood $U(A)$ of A there exists a neighborhood $V(A)$ of A such that if $x\in V(A)$ then $f^n(x)\in U(A)$ for any $n\in\mathcal{Z}^+$ and distance $\rho[f^n(x);(A)\to 0, n\to\infty]$. With one-to-one coupling we have to consider the two-dimensional map $F(x,y):\mathbb{R}^2\to\mathbb{R}^2$, given by equations

$$x_{n+1}=f(x_n)+d_1(y_n-x_n),$$
$$y_{n+1}=f(y_n)+d_2(x_n-y_n). \tag{1}$$

In this case the systems can be synchronized for some ranges of $d_{1,2}\in\mathbb{R}$, i.e., $|x_n-y_n|\to 0$ as $n\to\infty$ [2].

In the synchronized regime the dynamics of the coupled system (1) is restricted to a one-dimensional invariant subspace $x_n=y_n$, so the problem of synchronization of the chaotic systems can be understood as a problem of stability of a one-dimensional chaotic attractor A in two-dimensional phase space [1,2].

We define the basin of attraction $\beta(A)$ to be the set of points whose ω-limit set is contained in A. In Milnor's definition [4] of an attractor, the basin of attraction need not include the whole neighborhood of the attractor; we say that A is a weak Milnor attractor if $\beta(A)$ has positive Lebesgue measure. A riddled basin [1] that has been found to be typical for a certain class of dynamical systems with a one-dimensional invariant subspace [like $x_n=y_n$ in the example (1)] has positive Lebesgue measure but does not contain any neighborhood of the attractor. In this case basin of attraction $\beta(A)$ may be a fat fractal so that any neighborhood of the attractor intersects the basin with positive measure, but may also intersect the basin of another attractor with positive measure.

The dynamics of system (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold $x=y$ and is always positive. The second exponent characterizes evolution transverse to this manifold and it is called transversal. If the transversal Lyapunov exponent $\lambda_\perp$ is negative, the set A is an attractor at least in the *weak Milnor sense* [4], i.e., set A is a *weak Milnor attractor* if its basin of attraction $\beta(A)$ has positive Lebesgue measure in $\mathbb{R}^2$.

When the transversal Lyapunov exponent is still negative, but trajectories exist in the attractor A that are transversally repelling, A is a weak Milnor attractor with a *locally riddled* basin, (a set A is an attractor with locally riddled basin of attraction if there is a neighborhood U of A such that given any neighborhood V of any point in A, there is a set of points in $V\cap U$ of positive Lebesgue measure that leaves U in a finite time).

This riddling property has a local character. It describes behavior only in a sufficiently small neighborhood $U=U(A)$ and it gives no information about further behavior of the trajectories after they leave U. In the model under consideration, different situations relating to this global dynamics property take place. Two of the most common of them are the following [5].

(i) *Locally riddled basin.* After leaving neighborhood $U(A)$, almost all trajectories come back to U. Then, some fraction of them, after a finite number of iterations, leaves U again, and so on. Dynamics of such trajectories displays nonregular temporal ''bursting:'' a trajectory spends some time (usually a long time) near attractor A, than leaves; then, after some more time (usually short) it reenters neighborhood U.

(ii) *Globally riddled basin.* After leaving $U(A)$, a positive measure set of points goes to another attractor (or attractors) or to infinity. This other attractor may be, for instance, an attracting fixed point or an attracting cycle of chaotic two-dimensional sets.

Only in the globally riddled case can the basin of attraction $\beta(A)$ have the riddled structure of a fat fractal as a subset of $\mathbb{R}^2$. In this case the neighborhood of any point $(\bar{x},\bar{y})\in\beta(A)$ is filled by a positive Lebesgue measure set of points (x,y) that are attracted to another attractor (or attractors).

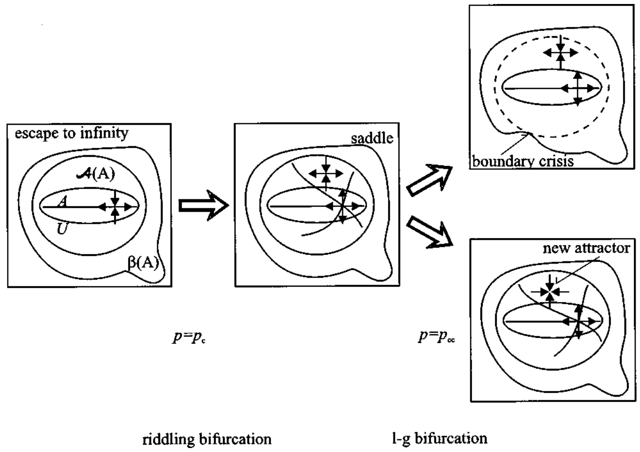

FIG. 1. *l-g* riddling bifurcations.

In this paper we identify and describe the bifurcations that appeared after riddling bifurcation and result in the transition from locally to globally riddled basins, and discuss the conditions for basins of attraction to be one or the other.

Riddling bifurcation occurs when some point x_p (unstable fixed or periodic point embedded in the chaotic attractor A in the invariant manifold) loses its transverse stability as a coupling parameter p passes though the critical value $p=p_c$. Lai *et al.* [7] showed that the loss of transverse stability may be induced by the collision at $p=p_c$ of two repellers r_+ and r_-, located symmetrically with respect to the invariant manifold, with the saddle at x_p. The result of this bifurcation was that a tongue opens at x_p and all preimages of x_p, allowing trajectories near the invariant manifold to escape from the U neighborhood of A for $p>p_c$, as shown in Fig. 1. Since preimages of x_p are dense in the invariant manifold, there is an infinite number of tongues, and a set of points leaving the U neighborhood has a positive Lebesgue measure. Note that this scenario is for the case where x_p undergoes (in transversal direction) a subcritical pitchfork bifurcation (corresponding to multiplicator crosses that are thought to be $+1$). A similar riddling mechanism works when x_p undergoes a subcritical period-doubling bifurcation (multiplicator crossing is thought to be -1) when the loss of transversal stability is induced by the collision at $p=p_c$ of only the one repeller r of the double period, the points of which are located symmetrically with respect to the invariant manifold with saddle at x_p. Moreover, there are completely different riddling scenarios in the case where saddle x_p undergoes a supercritical bifurcation in the transversal direction [5,7]. We have found here that the riddling scenarios for piecewise linear maps are quite different; at the moment that the transversal multiplicator of the saddle at x_p crosses $+1$ (or -1), there is no direct collision of the repellers (or single repeller) with the saddle. Before the bifurcation, these repellers (or single repeller) can exist but are located far from the invariant manifold. The bifurcation can change each of them to a saddle, or even into the attractor [8].

After leaving U a trajectory may or may not be captured by another attractor. If it is not captured it returns to U and the basin of attractor A is *locally riddled*. We will give evidence shortly that in order for a trajectory leaving the neighborhood of U to be captured by another attractor, one of the following conditions has to be fulfilled.

(1) A new attractor has to be born in one of the tongues inside the absorbing area $\mathcal{A}(A)$.

(2) The boundary of the immediate basin $\overline{\beta}(A)$ is broken as a result of the boundary crisis with the absorbing area $\mathcal{A}(A)$.

A new attractor is born in the repeller (saddle)-attractor bifurcation. After a riddling bifurcation a repeller (saddle) occurs due to the mechanism described in Ref. [7] and before transformation to the stable attractor repeller can undergo further bifurcations.

When one of the above conditions is fulfilled, say at $p=p_{cc}$, the basin of attractor A becomes globally riddled and all trajectories leaving neighborhood U converge to another attractor. This global bifurcation, which occurs at $p=p_{cc}$, we call a *local-global* (*l-g*) *riddling bifurcation*. It is said to be inner (outer) if condition (1) [Eq. (2)] is fulfilled.

The transition from asymptotically stable attractor A to globally riddled attractor A occurs after a sequence of local riddling bifurcations [6] and a *l-g* riddling bifurcation of either inner or outer type. These sequences of bifurcations are shown in Fig. 1.

To illustrate the described sequences of bifurcations, we consider the dynamics of a four-parameter family of two-dimensional piecewise linear noninvertible map F:

$$f_{l,p}(x_n)+d(y_n-x_n):$$

$$x_{n+1}=px_n+\frac{l}{2}\left(1-\frac{p}{l}\right)\left(\left|x_n+\frac{1}{l}\right|-\left|x_n-\frac{1}{l}\right|\right)+d_1(y_n-x_n), \tag{2}$$

$$f_{l,p}(y_n)+d(x_n-y_n):$$

$$y_{n+1}=py_n+\frac{l}{2}\left(1-\frac{p}{l}\right)\left(\left|y_n+\frac{1}{l}\right|-\left|y_n-\frac{1}{l}\right|\right)+d_2(x_n-y_n),$$

where $l,p,d_{1,2}\in\mathbb{R}$. Note that this system, which consists of two identical linearly coupled one-dimensional maps, is the generalization of the skew tent map. Chaotic attractors of skew tent maps have been considered in Refs. [8,9]. Two-dimensional map F, given by Eq. (1), is noninvertible as soon as one-dimensional map f is noninvertible in the sense that regions (points) exist that have two or more preimages. To characterize properties of system (1) we need some notation from the theory of noninvertible maps (for details see Ref. [10]). The basin of attraction $\beta(A)$ may be *connected* or *nonconnected*. If A is a connected attractor, the *immediate basin* $\overline{\beta}(A)$ is defined as the widest connected component of $\beta(A)$ containing A. Inside the immediate basin $\overline{\beta}(A)$ one can define an *absorbing area*. An area $\mathcal{A}(A)$ is said to be absorbing if (i) $F(\mathcal{A}(A))=\mathcal{A}(A)$, i.e., it is invariant with respect to F, and (ii) $\mathcal{A}(A)$ is attracting in the following strong sense: a neighborhood $U(\mathcal{A}(A))$ exists such that all its points are mapped inside $\mathcal{A}$ in a finite number of iterations. The boundary $\delta\mathcal{A}(A)$ is made up of segments of critical curves. Critical curves $l_k, k=0,1,\ldots$ are defined as successive images $l_k=F^k(l_0)$, $k=1,2,\ldots$ of the curve(s) l_0 that play the same role in two-dimensional maps as a critical (i.e., extremum) point (points) for one-dimensional maps. l_0 can be defined as a curve of merging preimages of F. It is clear

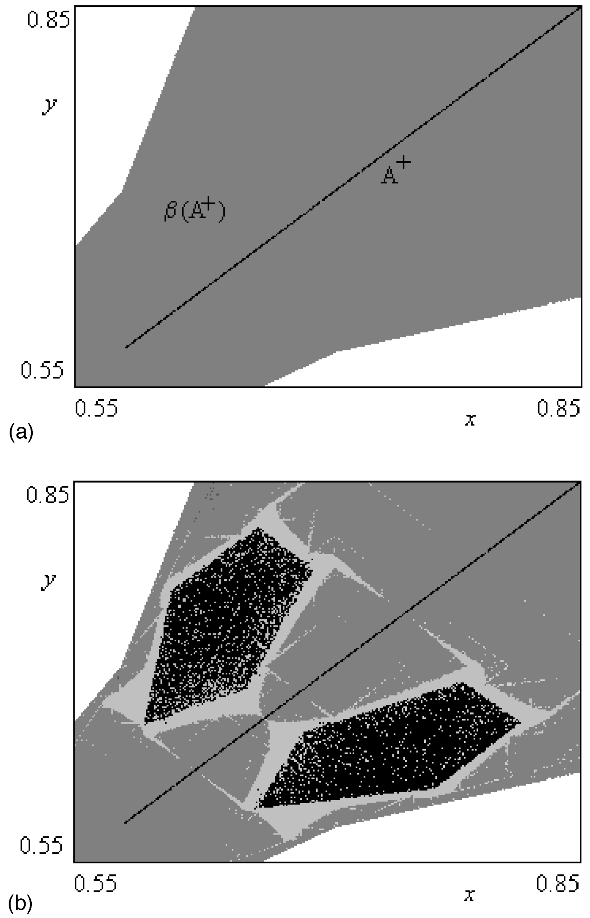

FIG. 2. Inner *l-g* riddling bifurcations of map (2). $l=\sqrt{2}$, $p=-\sqrt{2}$, (a) $d_1=d_2=-0.94$ locally riddled; (b) $d_1=d_2=-0.935$ globally riddled.

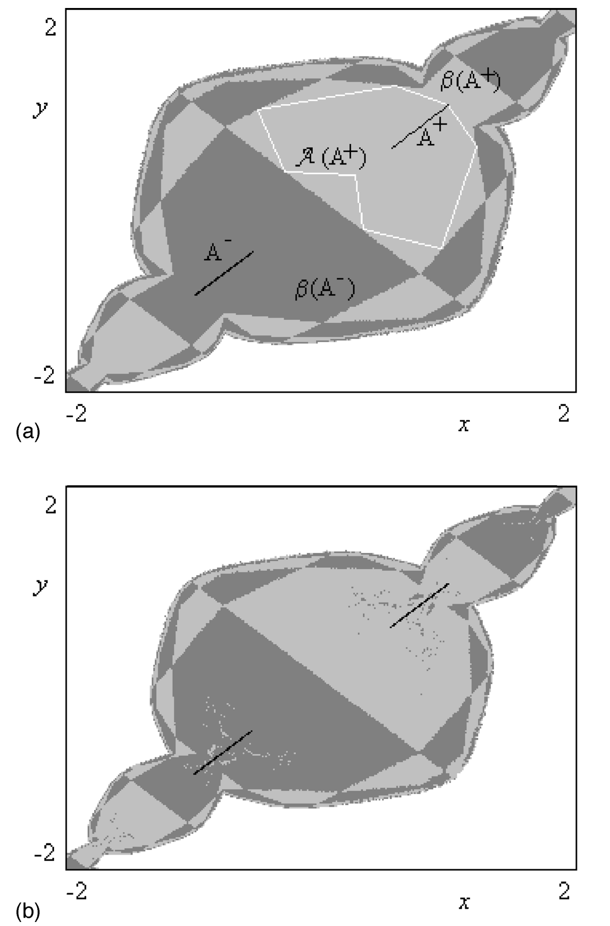

FIG. 3. Outer *l-g* riddling bifurcations of map (2). $l=1.3$, $p=-2$; (a) $d_1=d_2=0.674$ locally riddled; (b) $d_1=d_2=0.725$ globally riddled; attractors A^+ and A^- of the map (2) are (a) locally and (b) globally riddled by each other.

that l_0 belongs to the set $[(x,y)$; Jacobian of F, i.e., $|DF|$ vanishes or does not exist]. For a considered map (1) l_0 consists of two horizontal and two vertical lines: $l_0=\{x=\pm 1/l\,,\ y\in\mathbb{R}\}\cup\{x\in\mathbb{R},\ y=\pm 1/l\}$.

When $(l,p)\in\Pi=\{l>1,-l/(l-1)<p\leqslant -1\}$, the one-dimensional map $f_{l,p}$ has two symmetrical attractors $\Gamma^{(-)}\subset[-1,0]$ and $\Gamma^{(+)}\subset[0,1]$, which are cycles of 2^m chaotic intervals (so-called 2^m piece chaotic attractors). Depending on parameters l and p, m can be any positive integer. Denoting Π_m to be a subregion of Π where $\Gamma_m^{(\pm)}$ is a period-2^m cycle of chaotic intervals, the bifurcation curves for the transition $\Gamma_m^{(\pm)}\rightarrow\Gamma_{m+1}^{(\pm)}$ form the boundary between Π_m and Π_{m+1}.

For the map $F_{l,p}$, each set $A=A_m^{(\pm)}=\{x=y\in\Gamma_m^{(\pm)}\}$ is a one-dimensional chaotic invariant set that may or may not be an attractor in the plane (x,y). The various notions of attractors involve two kinds of properties: (i) that it attracts nearby trajectories and (ii) that it cannot be decomposed into smaller attractors. We shall concentrate on the first property, since A has dense trajectories everywhere and ''good'' invariant SBR measure μ (i.e., absolutely continuous with respect to the Lebegue measure) [11]. Thus the definitions given here should be completed by some minimality condition in order to be generally valid.

In comparison with the maps studied in Ref. [1] our map (2) has the advantage that, as shown in Ref. [5], conditions for the occurrence of locally riddled basins and conditions for attractor A to be asymptotically stable can be given analytically; it is therefore a useful test model for coupled chaotic systems.

Results of Ref. [5] guide the choice of parameter values, and we can distinguish two classes of *l-g* bifurcations, as demonstrated in Figs. 2 and 3.

An example of an inner type of *l-g* bifurcation is shown in Figs. 2(a) and 2(b) for $l=-p=\sqrt{2}$. As in the first type, before bifurcation [Fig. 2(a); $d_1=d_2=-0.94$] the basins of both attractors $A^{(+)}$ and $A^{(-)}$ are locally but not globally riddled (only attractor $A^{(+)}$ is shown). After the bifurcation, new attractors A_1 and A_2 are born [Fig. 2(b); $d_1=d_2=0.935$] in the neighborhood of $A^{(+)}$, and the basin of $A^{(+)}$ becomes globally riddled by the basins of these new attractors.

In Figs. 3(a) and 3(b), we show one example of outer *l-g* bifurcation for map (2). Before bifurcation in Fig. 3(a) ($l=1.3$, $p=-2$, and $d_1=d_2=0.6$) the basins of attractors $A^{(+)}$ and $A^{(-)}$ are locally riddled basins. Local riddling can be seen in Fig. 3(b) where the points leaving the neighborhood $(0.49,1.11)\times(0.49,1.11)$ of the attractor $A^{(+)}$ are shown in white. The boundary crisis between immediate basin of attraction $\bar{\beta}(A^{(+)})$ and absorbing area $\mathcal{A}(A)$ is indicated in

Fig. 3(a). After a bifurcation in Fig. 3(b) ($l=1.3$, $p=-2$, tors, are riddled by each other. and $d_1=d_2=0.725$) these basins are globally riddled and in fact, after a bifurcation basins of both $A^{(+)}$ and $A^{(-)}$ attrac

In contrast to the case of smooth maps [5,7] for the piecewise linear maps such as map (2), riddling bifurcation induced by pitchfork or period-doubling bifurcation of a cycle in the invariant manifold is always associated with a local-global bifurcation (born of the new attractor inside the absorbing area $\mathcal{A}(A)$. In this case $p_c=p_{cc}$ and we observe a direct transition from asymptotic stability to globally riddled basins. This fact has significant impact on the blowout bifurcation in the system but this problem is not addressed here [12,13].

In this paper we identified the global phenomena that can lead to a local-global riddling bifurcation. These might be either the breaking of an existing immediate basin boundary due to the boundary crisis with an absorbing area, permitting escape to a distant attractor, or the creation of a new attractor(s) within the existing basin but separate from the existing attractor; the basin(s) of the new attractor(s) riddles the basin of the initial attractor.

We stress that the model system Eq. (2) was used only for the purpose of illustrating the fundamental mechanism of *l-g* bifurcation. The observed properties of *l-g* bifurcations seem to be typical for a class of systems with lower-dimensional invariant manifolds, and are important for both the understanding and control of chaos synchronization.

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